Fall 2009 Math 245 Exam 1 Solutions

Exam scores: One quarter of the exam scores were below 71, one quarter between 71 and 76 (the median), one quarter between 76 and 83, and one quarter of the scores were above 83.

1. Carefully define the following terms:
   This problem tests the students' attention to detail and commitment to accurate definitions, which are very important in mathematics. The *inverse* of conditional proposition $p \to q$ is the proposition $\sim p \to \sim q$. *Modus Tollens* is a rule of deduction that concludes $\sim p$ from hypotheses $p \to q$ and $\sim q$. *Syntax* is a set of artificial rules about arrangements of otherwise meaningless symbols. *Semantics* is the interpretation of symbols to derive meaning about other things. A *tautology* is a compound proposition that is true for every possible combination of truth values of its components.

2. Simplify $\sim \forall x \exists y \exists z\ x + y \leq z$, to eliminate $\sim$.
   This problem tests negation of quantified propositions, as well as negation of simple arithmetic facts. $\exists x \forall y \forall z\ x + y > z$.

3. Simplify $\sim \big((p \vee \sim q) \wedge (r \vee (\sim s \wedge t))\big)$ as much as possible, so that no compound proposition is negated.
   This problem tests De Morgan's Law. $\sim (p \vee \sim q) \vee \sim (r \vee (\sim s \wedge t)) \equiv (\sim p \wedge \sim\sim q) \vee (\sim r \wedge \sim (\sim s \wedge t)) \equiv (\sim p \wedge q) \vee (\sim r \wedge (s \vee \sim t))$

4. Show that the proposition $s = (p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$ is a tautology.
   This problem tests complex, multistep, truth tables, and the meaning of "tautology". The last column of the table below proves that s is a tautology.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$p \wedge \sim q$	$\sim p \vee (p \wedge \sim q)$	s
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	F	T	T
F	F	T	T	F	F	T	T
							$\uparrow$

5. Use a truth table to determine whether this argument is valid:

$$\boxed{\begin{array}{r} p \\ p \to q \\ \sim q \vee r \\ \therefore\ \ r \end{array}}$$

   This problem tests the truth table method to determine validity of an argument.

p	q	r	$p \to q$	$\sim q$	$\sim q \vee r$	Reason for elimination
T	T	T	T	F	T	not eliminated
T	T	F	T	F	F	$\sim q \vee r$
T	F	T	F	T	T	$p \to q$
T	F	F	F	T	T	$p \to q$
F	T	T	T	F	T	p
F	T	F	T	F	F	p, $\sim q \vee r$
F	F	T	T	T	T	p
F	F	F	T	T	T	p

   All remaining rows (the first one) have r true, hence the argument is valid. Note: it is important to justify why each row is being eliminated.

6. Construct the truth table for the proposition $(p \to r) \leftrightarrow (q \to r)$.
This problem tests complex truth tables involving conditionals and biconditionals.

p	q	r	$p \to r$	$q \to r$	$(p \to r) \leftrightarrow (q \to r)$
T	T	T	T	T	T
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

7. Find a proposition using only combinations of $p, q, \downarrow$, that is logically equivalent to $p \wedge q$.
This problem tests understanding of compound NOR propositions. The answer is given in the last column, i.e. $p \wedge q \equiv (p \downarrow p) \downarrow (q \downarrow q)$.

p	q	$p \downarrow p$	$q \downarrow q$	$(p \downarrow p) \downarrow (q \downarrow q)$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	F

8. Convert the number BAD_{16} to base 2 and to base 10.
This problem tests base conversion. Conversion to base 2 can be done in groups of four bits, giving $1011\ 1010\ 1101_2 = 101110101101_2$. Conversion to base 10 is done by: $11 \cdot 16^2 + 10 \cdot 16 + 13 = 2989_{10}$.

9. Fill in the missing justifications, including line numbers, for the following proof.
This problem tests application of the rules of logic to construct a valid proof.

1.	$(p \vee q) \vee r$	hypothesis
2.	$s \to c$	hypothesis
3.	$p \vee r \to s$	hypothesis
4.	$\sim s$	rule of contradiction, applied to 2.
5.	$\sim (p \vee r)$	modus tollens, applied to 3. and 4.
6.	$\sim p \wedge \sim r$	De Morgan's Law, applied to 5.
7.	$\sim r$	conjunctive simplification, applied to 6.
8.	$p \vee q$	disjunctive syllogism, applied to 1. and 7.
9.	$\sim p$	conjunctive simplification, applied to 6.
10.	$\therefore q$	disjunctive syllogism, applied to 8. and 9.

10. All bizzles are azzles, but not all azzles are bizzles. There is at least one cozzle that is also a bizzle. There is at least one dizzle that is not a bizzle. All cozzles are dizzles. Must there be a dizzle that is also an azzle?
This problem tests semantic translation into symbols, as well as interpretation of quantified propositions. For universe element x, let $A(x)$ be the proposition that x is an azzle, etc.
1. $\forall x \in U, B(x) \to A(x)$
2. $\exists x \in U, A(x) \wedge \sim B(x)$ *not needed for proof*
3. $\exists x \in U, C(x) \wedge B(x)$
4. $\exists x \in U, D(x) \wedge \sim B(x)$ *not needed for proof*
5. $\forall x \in U, C(x) \to D(x)$

By 3., there is some universe element, call it George, that is both a bizzle and a cozzle. By 1., George is also an azzle. By 5., George is also a dizzle. Hence George is an azzle and a dizzle, so the answer is 'yes'.